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ON THREE FORGOTTEN RESULTS OF S.KREIN, N. BOGOLYUBOV AND V. GURARI WITH APPLICATIONS TO BERNSTEIN OPERATORS

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1. Introduction

The results of M. Frechet in 1934 about the largesteigenvalue of a stochastic matrix [6] attracted attentionto positive linear operators with norm 1. The study of compact linearoperators with stochastic kernel by N.M. Krylov and N.N. Bogolyubovduring the 1930's ([9], [10]) was generalized by S. Krein and N.N. Bogolyubov a decade later in [2]. These results contributed to the 1968 paper [8] of M. Krasnoselski in which the problem of determining minimal-normshape-preserving projections was present. Unfortunately, many of these papers are practically unknown, as they were published in the Ukrainian language. On the other hand, recent developments in thetheory of minimal shape-preserving projections have been made using methodsthat are independent of Krasnoselski's work (see [3] and [11]. In this paper, we attemptto connect these two directions by studying the (so called) *Bernstein operators*.

Let $C[0,1]$ be the space of real-valued continuous functions and $\mathbb{N}$ be the set of positive integers. Let $n \in \mathbb{N}$. By the Bernstein polynomials we mean the particular family of n-th polynomials given by the formula

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} f(k/n) x^k (1-x)^{n-k} \qquad (1)$$

where $0 \leq x \leq 1$ and $f \in C[0,1]$. The operator which send seach $f \in C[0,1]$ to the Bernstein polynomial $B_n(f,x)$ will becalled the *Bernstein operator* (or B_n for brevity). See [13] for details concerning Bernstein polynomials.

Thus B_n is a linear operator mapping $C[0,1]$ onto the subspace $\mathbb{P}_n$ of polynomials of degree less than or equal to n. Itis well-known that the polynomials $x^k(1-x)^{n-k}$, $k = 0,\dots,n$ form a basis for the space $\mathbb{P}_n$. Additional well-knowncharacteristics of the Bernstein polynomials translate immediatelyinto properties enjoyed by the Bernstein operators; in the following, we list (as lemmas) some of these properties.

LEMMA 1. *Let $e_\nu(t) = t^\nu$, where $\nu \geq 0$. Let $n \in \mathbb{N}$. Then*

$$B_n(e_0) = e_0,\ B_n(e_1) = e_1, \text{ and } B_n(e_2)(x) = e_2(x) + \frac{1}{n}x(1-x).$$

LEMMA 2. *Let $f \in C[0,1]$ such that $f(x) \neq cx + b$. Then for each $n \in \mathbb{N}$, $B_n(f) \neq f$.*

LEMMA 3. *For each $f \in C[0,1]$, we have*

$$\|B_n(f)\| \leq \|f\|$$

where $\|f\| = \sup_{t\in[0,1]} |f(t)|$. Moreover, $\|B_n\| = 1$ for each $n \in \mathbb{N}$.

2. Forgotten Results

We begin with the 1947 result by Bogolyubov and Krein. Let T be acompact linear operator with norm 1 in Banach space E.

THEOREM 1 ([2], see also statistical ergodic theorems in [4]). *The sequence of operators T_j, where $j \in \mathbb{N}$ and*

$$T_j = \frac{1}{j}\sum_{n=1}^{j} T^n \tag{2}$$

converge in the operator norm to the compact linear operator T_∞ such that

$$TT_\infty = T_\infty T = T_\infty^2 = T_\infty \tag{3}$$

DEFINITION 1. *Let E be a partially ordered Banach space (with the partial orderingdenoted by $>$). If $K \subset E$ is aconvex set, closed under nonnegative scalar multiplication, then wecall K a cone of E. Let $P : E \to E_0$ be a projection ontosubspace E_0. We say P is a* shape-preserving *projection if*

$$PK \subset K. \tag{4}$$

In this setting, the elements of cone K are said to *have shape*(in the sense of K).The positive elements of E are those x such that $x \geq 0$. Notethat theset of all positive element of E forms a cone. We say operator T is a *positive operator* if $Tx \geq 0$ whenever $x \geq 0$. An element $x \in E$ is an *invariant element* of T if $Tx = x$.

In [2] we also find the following result:

THEOREM 2. *If T is a positive compact linear operator in the partially ordered Banach space E then the set of all invariant elements of T form afinite-dimensional subspace E_0 of E with shape-preservingprojection $P_0 = T_\infty : E \to E_0$ (if $\dim E_0 > 0$). Here theelements with shape are the positive elements of E.*

The following corollary is an immediate consequence of applying the above results to the Bernstein operators.

COROLLARY 1. *Fix $n \in \mathbb{N}$ and let B denote the n-th degree Bernstein operator (i.e. $B = B_n$). Then*

$$P_\infty = \lim_{j\to\infty} B_j$$

is a shape-preserving projection onto 2-dimensional subspace E_0 (generated bye$_0$ and e_1) and $\|P_\infty\| = 1$, where

$$B_j = \frac{1}{j}\sum_{n=0}^{j} B^n.$$

Proof. The shape in this theorem is positivity. The existence of P_∞ follows from Theorem 1; the fact that P_0 isshape-preserving onto the span of e_0 and e_1 follows from Lemma 1and Theorem 2. The norm equal 1 claim is a consequence of Lemma 3. ■

THEOREM 3 ([8]). *Let E_1 be an s-dimensional ($s > 1$) subspace of $C[0,1]$. Let*

$$U = \{x \in C[0,1] \mid \|x\| \leq 1\},$$

the unit ball of $C[0,1]$. There exists a norm 1 projection $P : C[0,1] \to E_1$ if and only if the section $U \cap E$ is a polytope with $2s$ faces (where the dimension of each face is $s-1$).

From Theorem 3 and Corollary 1 we obtain:

COROLLARY 2. *The unit sphere of subspace E_0 (from Corollary 1) is a centrally symmetric quadrilateral.*

3. Applications and Extensions

We apply and extend the results of the previous section in the following notes.

NOTE 1. Let B_n^* denote the canonical dual operator to B_n; i.e., foreach $\varphi \in (C[0,1])^*$ we have $(B_n^*(\varphi))(f) = \varphi(B_n(f))$. Because $C[0,1]$ is a partially ordered vector space,there exists a norm-1 element, u, such that for each non-zero $h \in C[0,1]$ the set

$$A_u = \{t > 0 \mid -tu < h < tu\}$$

has $\inf A_u \neq 0$. We use u to define the set μ^* asfollows:

$$\mu^* = \{\varphi \in (C[0,1])^* \mid \varphi > 0, \varphi(u) = 1, \varphi(f) = B_n^*(\varphi(f))\}.$$

Note that we say non-zero functional $\varphi > 0$ whenever $\varphi(f) \geq 0$ for$f \geq 0$.

COROLLARY 3. *μ^* is a simplex.*

This result is a consequence of the Theorem 2 from [2]. Wecan also prove it directly, using properties of B_n. Indeed, sinceB_n is a positive compact linear operator, it follows that B_n^* is as well (see for example [5]). We also have $\|B_n^*\| = \|B_n\| = 1$. Consequently we have that the image of $(C[0,1])^*$ under B_n^* is a finite-dimensional subspace D_n (see [5]). Let $D \subset D_n$ be the set of all functionals from $(C[0,1])^*$ which areinvariant elements of B_n^*. Select for D a basis so that thenonnegative elements D^+ of D have positive coordinates (thusD^+ is the positive orthant of D_n). Thus we see that μ^* is simply the intersection of D^+ with the hyperplane of allfunctionals ψ such $\psi(u) = 1$ and, therefore, is a simplex.

NOTE 2. In Corollary 1 the dimension of the subspace of invariant elements of B_n was 2. Other dimensions of $E_0 = T_\infty(C[0,1])$ can be obtained using different positive compactoperators acting on $C[0,1]$. For example, if F is a continuousfunction on the square $[0,1;0,1]$ and T is the operator defined by

$$Tx(s) = \int_0^1 F(s,t)x(t)\,dt$$

where $x \in C[0,1]$, then T will be a compact operator. It is wellknown ([5]) that

$$\|T\| = \max_{0 \leq s \leq 1} \int_0^1 |F(s,t)|\,dt$$

if we require $F(s,t) \geq 0$ then we have $\|T\| = 1$. In this case wecall F a *stochastic kernel.* Furthermore, if $F \equiv 1$ then $\|T\| = 1$ and $Tf(s)$ is constant for each $f \in C[0,1]$. Hence $\dim E_0 = 1$ where $E_0 = T_\infty(C[0,1])$. From here, for example, wecan use tensor products (see[12]), to build shape-preserving projections onto arbitraryfinite-dimensional subspaces.

NOTE 3. In this paper we considered the Bernstein operators. Interesting results can be obtained by studying analogous operators. For example,from [1] we find the family of operators

$$(Q_n f)(x) = \sum_{m=0}^{n} \left[f\left(\frac{m}{n}\right) - \frac{x(1-x)}{2n} f''\left(\frac{m}{n}\right) \right] \binom{n}{m} x^m (1-x)^{n-m}$$

and from [14] we find the Kantorovich polynomials given by

$$K_n(f,x) = B_n^{'}(F,x)$$

where f is an integrable function on $[0,1]$ and $F(x) = \int_0^t f(t)\,dt$.

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