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THE CHARACTERISATION OF VAN KAMPEN-FLORES COMPLEXES BY MEANS OF SYSTEM OF DIOPANTINE EQUATIONS

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In 1967, V.Rokhlin asked to describe all the t-dimensional pairs of Van Kampen-Flores complexes having the same homology groups.In this paper we give a complete characterization of such KF-complexes.

Let C^n_{2n+3} be the n-dimensional skeleton (the complete n-complex) of the 2n + 2-dimensional simplex.

Denote by "$\vee$" the join of two topological spaces X and Y, that is, $X \vee Y = (KX \times Y) \cup (X \times KY)$, where $KX = (X \times [0,1]) / \{(x,1) \cong (x',1)\}$ is the cone on X and KY is the cone on Y.

Denote by $B(p_1, \ldots, p_r)$ the join of complexes of the form

$$B(p_1, \ldots, p_r) = \bigvee_{i=0}^{r-1} \bigvee_{p_{i+1}} C^i_{2i+3}, \tag{1}$$

where p_i are non-negative integers for i = 1, ..., r, with $p_r > 0$ and

$$k = \left(\sum_{i=1}^{r} i p_i \right) - 1 \tag{2}$$

is the dimension of $B(p_1, \ldots, p_r)$.

We refer to the complex B^k of the form (1) as a Van Kampen-Flores complex, or KF-complex. In 1932-1933 Van Kampen and Flores gave two examples of n-dimensional polyhedra which are not embeddable in R^{2n} (see [1]-[3]). More precisely, $B(n+1)$ was an example of Flores and $B(0, \ldots, 0, n)$ was an example of Van Kampen. For n = 1, a Theorem of Kuratowski (see [3]) shows also that the two KF-complexes $B(2)$ and $B(0, 1)$ are the minimal topological non-homeomorphic polyhedra which are not embeddable in R^2. So far, no generalization for $n \geq 2$ of the above result of Kuratowski has been given. In particular,

we do not know if the n-dimensional *KF*-complexes are the only minimal topological polyhedra of dimension n which are not embeddable in R^{2n}.
In 1966, V. Rokhlin conjectured that if $B(p_1, \ldots, p_n)$ and $B(q_1, \ldots, q_r)$ are homeomorphic then $n = r$ and $p_i = q_i$, for $i = 1, \ldots, n$. The above conjecture has not yet been settled.
It is well-known that if two n-dimensional polyhedra are homeomorphic, then they have the same homology groups. The *KF*-complex has the property that $B(p_1, \ldots, p_n)$ is homotopically equivalent to a t-dimensional buquet of spheres where $t = \sum_{i=1}^{n} ip_i - 1$. In an attempt to prove this conjecture, it is important to understand when two *KF*-complexes have the same homology groups. In [8], one of us found that even for the dimension $k = 16$ one can find two different *KF*-complexes $B^k(p_1,\ldots,p_n)$ and $B^k(q_1,\ldots,g_r)$ with $r < n$ which have the same topological type. In 1967, V. Rokhlin asked to describe all the t-dimensional pairs of *KF*-complexes having the same homology groups. Since the rank of the homology group of $B(p_1,\ldots,p_n)$ is $\prod_{i=1}^{n}\binom{2i}{i}^{p_i}$, Rokhlin's question is equivalent to finding natural numbers $t \in N$ such that

$$t = \sum_{i=1}^{n} ip_i - 1 = \sum_{j=1}^{r} jq_j - 1, \tag{3}$$

and

$$\prod_{i=1}^{n}\binom{2i}{i}^{p_i} = \prod_{j=1}^{r}\binom{2j}{j}^{q_j}. \tag{4}$$

In this paper, we give a complete characterization of such *KF*-complexes. The proof is based on the existence of solutions of the above system of diophantine equations (3) and (4) which can have, in our opinion, independent interest.
I. Consider the following system of diophantine equations:

$$\sum_{i=1}^{r} iq_i = \sum_{i=1}^{n} ip_i \tag{5}$$

$$\prod_{i=1}^{r}\binom{2i}{i}^{p_i} = \prod_{i=1}^{r}\binom{2i}{i}^{q_i} \tag{6}$$

having $r < n$ and $p_n \neq 0$. By making the substitutions

$$x_i = \begin{cases} p_i - q_i & \text{for } i = 1,\ldots,r, \\ p_i & \text{for } i = r+1,\ldots,n \end{cases} \tag{7}$$

the system becomes

$$\sum_{i=1}^{n} ix_i = 0 \tag{8}$$

$$\prod_{i=1}^{n} \binom{2i}{i}^{x_i} = 1 \tag{9}$$

where $x_n \neq 0$. Our result is

Theorem 1.
The system of diophantine equations (8) *and* (9) *has an integer solution if and only if* $n \neq 5$ *or* $2n-1$ *is not prime.*

Proof. It is clear that the system has no solutions for $n = 1$ or $n = 2$. Indeed, for $n = 1$, equation (8) implies $x_1 = 0$, while for $n = 2$ equation (9) implies

$$1 = \binom{2}{1}^{x_1} \binom{4}{2}^{x_2} = 2^{x_1+x_2} 3^{x_2},$$

which leads to $x_1 = x_2 = 0$. From now on, we assume that $n \geq 3$.
For every positive integer i and every prime number p let

$$e_{pi} = \text{ord}_p\left(\binom{2i}{i}\right). \tag{10}$$

It is well-known that if k is any positive integer and p is any prime, then

$$\text{ord}_p(k!) = \frac{k - \sigma_p(k)}{p-1},$$

where $\sigma_p(k)$ is the sum of the digits of k written in base p. In particular, formula (10) implies that

$$e_{p,i} = \frac{2\sigma_p(i) - \sigma_p(2i)}{p-1}. \tag{11}$$

Notice that every prime divisor of

$$\prod_{i=1}^{n} \binom{2i}{i}$$

is less than or equal to $2n$. With these remarks, it follows that equation (9) is equivalent to the linear homogeneous system

$$\sum_{i=1}^{n} e_{p,i} x_i = 0 \qquad \text{for all } p \leq 2n \tag{12}$$

When $2n - 1 = q$ is a prime, then $e_{i,q} = 0$ for all $i < n$ but $e_{n,q} = 1$. Hence, the last equation (12) in this case is $x_n = 0$ which contradicts our assumption.
Hence, a necessary condition for the existence of a solution of system of equations (8) and (9) with $x_n \neq 0$ is that $2n - 1$ is not prime.

From now on we assume that $2n - 1$ is not prime.

Notice now that the system of equations (8) and (9) is equivalent to the homogeneous linear system
of equations (8) and (12). This system has exactly $\pi(2n)+1$ equations (there are precisely $\pi(2n)$ prime numbers less than or equal to $2n$ and we add one for the equation (8)) and n unknowns. We
first notice that

$$\pi(2n)+1<n$$

for all $n\geq 8$. Indeed, if $n\geq 11$ and $p<2n$ is prime, then

$$p\in\{1,2,\ldots,2n\}-(\{2,6,,8,\ldots,2n\}\cup\{9,15,21\}),$$

therefore $\pi(2n)\leq(2n-(n-1+3)=n-2$ for $n\geq 11)$. One can check directly that this inequality holds for $n\in(8,9,10)$ too.
When $3\leq n\leq 7$ the number 2n – 1 is always prime except for n = 5. We will show that the system of equations (8) and (12) has no solutions for n = 5. Indeed, in this case the 5 equations

$$\begin{cases} x_1+2x_2+3x_3+4x_4+5x_5=0, \\ x_1+x_2+2x_3+x_4+2x_5=0, \\ x_2+2x_5=0, \\ x_3+x_4=0, \\ x_4+x_5=0, \end{cases} \qquad (13)$$

where the last four equations are system (12) for $p\in\{2,3,5,7\}$. It is easily seen that system (13) is non-singular, hence $x_i = 0$ for all i = 1, ..., 5.
This proves the necessity. For the sufficience, assume now that $n\geq 3$, $n\neq 5$ and 2n – 1 is not prime. In this case, $n\geq 8$. Hence, $\pi(2n)+1<n$, therefore the system of equations (8) and (12) has more equations than solutions. In this case, this system has a non-trivial integer solution $(x_1,\ldots,x_n)$. We have to show that one can find a solution with $x_n\neq 0$. In order to do so, it suffices to show that one can choose a subset $I\in\{1,\ldots,n-1\}$ of cardinality $\pi(2n)+1$ such that the corresponding minor is non-zero. We claim that we can choose

$$I=\left\{1,5,\frac{p+1}{2} \text{ for all odd primes } p\leq 2n\right\}$$

Notice that since $n\geq 8$ and $2n$ – 1 is not prime, it follows that I does not contain n and I has precisely $\pi(2n)+1$ elements $(5=(9+1)/2$ and 9 is not prime$)$. If one writes the upper left six-by-six corner of the corresponding minor, this is precisely

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \ldots \\ 1 & 1 & 2 & 1 & 2 & 2 & \ldots \\ 0 & 1 & 0 & 0 & 2 & 1 & \ldots \\ 0 & 0 & 1 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 1 & 1 & 1 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 1 & \ldots \end{array} \qquad (14)$$

Because of the occurence of 0 in position (6,5) of the above matrix, it is easily seen that one can perform elementary operations on these 6 rows is such a way as to bring this six-by-six corner to an upper triangular form with five of the elements on the main diagonal equal to 1 and the sixth element equal to -1. For the remaining part of the minor, notice that if $p \geq 13$ is any prime and $k < (p+1)/2$ is any positive integer, then

$$e_{p,k} = 0,$$

while

$$e_{p,(p+1)/2} = 1.$$

Hence, the minor formed with elements of I is (up to a few transformations involving only the first six rows) in standard upper triangular form with all elements on the main diagonal equal to 1 except for one of them which is equal to -1. This shows that one can solve the homogeneous system of equations formed with the equations (8) and (12) having the unknowns $(x_i)_{i \in I}$ as principal unknowns and the other ones (among which is x_n too) as parameters. Each rational choice of the parameters will give (via Kramer's rule) a rational non-zero solution of the given system, which in turn can be lifted to an integer solution by multiplying all the unknowns with an appropriate integer (the common denominator of all the rational components of the given solution).

II. Theorem 2.

Let $B_1 = B^k(p_1, \ldots, p_n)$ be a KF-*complex with $n \geq 6$. Then, there exists another* KF-*complex $B_2 = B^k(q_1, \ldots, q_r)$ with $r < n$ having the same topological type as B_1 if and only if $2n-1$ is not prime.*

Proof. We know (see corollary 1 in [7]) that the skeleton C_{2i+1}^{i-1} has the homotopy type of a wedge of $\binom{2i}{i}$-copies of the i-dimensional sphere S^i and that

$$H_0(B_1) = Z,$$

$$H_1(B_1) = \ldots = H_{k-1}(B_1) = 0 \qquad \text{for } k \geq 2$$

and

$$H_k(B_1) = \underbrace{Z \oplus Z \oplus \ldots \oplus Z}_{\prod_{i=1}^{n} \binom{2i}{i}^{p_i}} \qquad (15)$$

The assertion of Theorem 2 follows now from Theorem 1.

Remark. From Theorem 2, we get that the first four numbers n for which there exists a different KF-complex $B_2 = B(q_1, \ldots, q_r)$ with $r < n$ of the same topological type as B_1 are $n = 8, 11, 13, 14$ (compare with remark 1 in [8]).

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References

1. **Van Kampen E.R.** Komplexe in Euklidischen Raumen//*Abh. Math. Sem. Uni. Hamburg. 1932. № 9 . P. 72-78 und P. 152-153.*
2. **Flores A.I.** Uber die Existenz n-dimensionaler Komplexe, die nicht in den .R^{2n} topologisch einbettbar sind//*Ergebn. Math. Kolloq. 1932/33. № 5. P.17-24.*
3. **Kuratowski K.** Sur Ie probleme des courbes gauches-en topologie// *Fund. Math. 1930. № 15. P. 271-283.*
4. **Grünbaum V.** Graphs and complexes//*Lecture Notes. Univ. of Washington. 1967.*
5. **Odyniec W.P.** A generalization of Van Kampen and A. Flores examples// *Thesis of M. Sc., Leningrad Univ. 1967. P. 1-19 (in Russian).*
6. **Zaks J.** On minimality property of complexes//*Proc. Amer. Math. Soc. 1969. № 20. 1969. P. 439-444.*
7. **Grünbaum B.** Embeddings of Simplicial Complexes//*Comment. Math. Helvet. 1969. № 44. P. 502-513.*
8. **Odyniec W.P.** On the Rohlin hypothesis//Zeszyty nauk. WSP w Bydgoszczy, Probl. Mathem. z. 8 . 1986. P. 5-16.

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