ПРИКЛАДНАЯ МАТЕМАТИКА И МЕХАНИКА

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ON FACTORIZATION OF TRIANGLE MATRIX FUNCTIONS

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The paper is devoted to an analysis of the efficient factorization method for triangular matrix-functions of arbitrary order, which generalizes G. N. Chebotarev's method. Results are illustrated by examples.
Keywords: matrix-functions factorization, triangular matrices, continuous fractions.

1. Introduction

Let Γ be a simple smooth closed curve on the complex plane $\mathbb{C}$ dividing $\mathbb{C}$ into two domains $D^+ \ni 0$ and $D^- \ni \infty$. By the factorization of a non-singular continuous complex-valued matrix-function $G \in (\mathscr{C}(\Gamma))^{n\times n}$ it is understood the determination of two matrices $G^\pm$ analytic in $D^\pm$, respectively, together with their inverses $(G^\pm)^{-1}$, and of the diagonal matrix

$$\Lambda(t) = diag\left\{t^{\kappa_1}, \ldots, t^{\kappa_n}\right\}, \quad \kappa_1, \ldots, \kappa_n \in \mathbb{Z},$$

such that the following representation holds on Γ:

$$G(t) = G^+(t)\Lambda(t)G^-(t), \quad t \in \Gamma. \tag{1}$$

The representation (1) is called the *left (continuous or standard) factorization*. Interchanging and we arrive at the *right (continuous or standard) factorization*. If the left (right) factorization exists, then the integer numbers $\kappa_1, \ldots, \kappa_n \in \mathbb{Z}$ are determined uniquely up to their order (thus, one can always suppose $\kappa_1 \geq \ldots \geq \kappa_n$). These numbers are called *partial indices*. The factors G^+, G^- in (1) are determined non-uniquely (they can be found up to multiplying on special non-singular polynomial matrices, see [7]).

Initially, the factorization problem is linked to B.Riemann or, more precisely, with two problems formulated by him, known as the *Riemann boundary value problem* (or *Riemann-Hilbert boundary value problem*, see [4]), and the *Riemann monodromy problem* (or the *21st Hilbert problem*, or the *Riemann-Hilbert problem*, see [2]). In the present day, the factorization problem is interesting due to its connections to notable mathematical problems (vector-matrix boundary value problems, systems of singular integral equations, the Wiener-Hopf and other convolution type equations, the Riemann-Hilbert problem, classification of vector bundles on the Riemann sphere, nonlinear evolution equations, the Toeplitz operators, etc), as well as to applied problems (elasticity and elasto-plasticity, radiation and neutron transport, wave diffraction, fracture mechanics, geomechanics, signal processing, financial mathematics, etc, see, e.g. [5,6]). Sometimes the factorization problem is called the Wiener-Hopf factorization, since it is connected with the Wiener-Hopf technique developed initially for the study of the Wiener-Hopf integral equation (see, e.g. [6]).

In spite of the extended interest to the factorization problem and its rather developed theory, the constructive direction of this branch is far from completeness (see, the recent survey on constructive methods of factorization [10]). For special classes of matrix-functions there exist several important approaches describing determination of partial indices and construction of factors. Among others (see [10]) we can mention here the paper by G.N.Chebotarev [3] on factorization of triangular matrix-functions of the second order, and the results by V.M.Adukov [1] presenting an algorithm of the constructive factorization of meromorphic matrix-functions. Chebotarev's method was recently generalized for triangular matrix-functions of arbitrary order [9]. Here we briefly describe the later approach illustrating it by certain examples. Without loss of generality we take the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ as the curve Γ in these examples.

2. Factorization of triangular matrix-functions of arbitrary order

The central aim of [9] is to provide an inductive approach and to reduce to factorization of the matrix-functions of higher order to the factorization of lower order of matrices. The basic tools making this method efficient are two statements.

Lemma 1. *([3]) Let us consider a 2-nd order non-singular triangular* *matrix-function*

$$A(t) = \begin{pmatrix} \zeta_1(t) & 0 \\ a(t) & \zeta_2(t) \end{pmatrix}.$$

Let $\kappa_j = ind_\Gamma\, \zeta_j(t)$ *and let* $x_j^\pm(z)$ *be canonical functions for the homogeneous Riemann boundary value problems with coefficients* $\zeta_j(t), j = 1, 2$*, respectively (see [3]). Let* $\mu \geq 1$ *be the order at infinity of the following function*

$$\phi^\pm(z) = \frac{1}{2\pi i} \int\limits_\Gamma \frac{a(\tau)x_1^-(\tau)d\tau}{\tau - z}, \quad z \in D^\pm.$$

If $\kappa_1 \leq \kappa_2 + \mu$*, then matrix-function possesses factorization*

$$A(t) = X^+(t) \begin{pmatrix} t^{\kappa_1} & 0 \\ 0 & t^{\kappa_2} \end{pmatrix} X^-(t), \quad X^\pm(t) = \begin{pmatrix} x_1^\pm & 0 \\ x_2^\pm \phi^\pm & x_2^\pm \end{pmatrix},$$

with partial indices κ_1, κ_2*.*

If $\kappa_1 > \kappa_2 + \mu$*, then the function* $\frac{1}{\phi^-(z)}$ *is represented in the continued fraction*

$$\frac{1}{\phi^-(z)} = q^{\gamma_0}(z) + \frac{1}{q^{\gamma_1}(z) + \frac{1}{q^{\gamma_2}(z) + \dots}},$$

where $q^{\gamma_i}(z)$ *are polynomials of order* γ_i*,* $\gamma_0 = \mu$*. Denote* $\mu_1 = \gamma_0 + \gamma_1$*,* $\mu_2 = \gamma_0 + \gamma_1 + \gamma_2$*, ... If* $\mu_{i-1} + \mu_i < \kappa_1 - \kappa_2$*, but* $\mu_i + \mu_{i+1} \geq \kappa_1 - \kappa_2$*, then the partial indices of the matrix* $A(t)$ *are equal* $\kappa_1 - \mu_i$*,* $\kappa_2 + \mu_i$*, and the factors are constructed by using representation of the function* $\frac{1}{\phi^-(z)}$ *and elementary transformations of the columns.*

Lemma 2. *([9]) Let* $B(t), t \in \Gamma$*, be a non-singular Hölder continuous square matrix-function of the order* n *having the following form:*

$$B(t) = \begin{pmatrix} A(t) & \mathbf{0} \\ b_1(t) \dots b_{n-1}(t) & c(t) \end{pmatrix}, \quad \mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2}$$

Let the non-singular matrix-function $A(t)$ *of the order* $n - 1$ *admits factorization*

$$A(t) = A^+(t)\Lambda(t)A^-(t) = A^+(t) diag\left\{t^{\kappa_1}, \dots, t^{\kappa_{n-1}}\right\} A^-(t).$$

Then the matrix-function $B(t)$ *possesses factorization if the following matrix does:*

$$\begin{pmatrix} \Lambda(t) & \mathbf{0} \\ \mathbf{b}(t)|\mathbf{Y}_1(t) \dots \mathbf{b}(t)|\mathbf{Y}_{n-1}(t) & c(t) \end{pmatrix}, \tag{3}$$

where $\mathbf{Y}_j(t)$ *is the* j*-th column of the matrix* $Y(t) = (A^-(t))^{-1}$,

$$\mathbf{b}(t)|\mathbf{Y}_j(t) = \sum_{k=1}^{n-1} b_k(t) Y_{kj}(t).$$

Example 1. *Let us illustrate the reduction of the factorization problem of the matrix-function of the form (2) to the factorization of the triangular matrix function of the form (3). Consider the matrix-function*

$$B(t) = \begin{pmatrix} & A(t) & \mathbf{0} \\ \frac{1}{t+2} & \frac{1}{t+2} & \frac{t-2}{t+3}\,\frac{3t+2}{3t-1} \end{pmatrix},$$

where $A(t)$ *is the second order non-singular square matrix*

$$A(t) = \begin{pmatrix} \frac{t^3 - 3t^2 + 1}{t-3} & \frac{-3t^4 + 7t^3 + 6t^2 + 3t - 1}{3t^2 - 7t - 6} \\ \frac{t^3 - 6t^2 + 1}{t-2} & \frac{-9t^4 + 12t^3 + 12t^2 + 3t - 1}{3t^2 - 4t - 4} \end{pmatrix}.$$

The matrix-function $A(t)$ possesses the following (bounded) factorization

$$A(t) = A^+(t)\Lambda(t)A^-(t),$$

where

$$A^+(t) = \begin{pmatrix} 1 & \frac{1}{t-3} \\ 3 & \frac{1}{t-2} \end{pmatrix}, \ \Lambda(t) = \begin{pmatrix} t^2 & 0 \\ 0 & 1 \end{pmatrix}, \ A^-(t) = \begin{pmatrix} 1 & -1 \\ 1 & \frac{3t-1}{3t+2} \end{pmatrix}.$$

The corresponding matrix $Y(t) = (A^-(t))^{-1}$ can be found directly

$$Y(t) = (A^-(t))^{-1} = \begin{pmatrix} \frac{3t-1}{6t+1} & \frac{3t+2}{6t+1} \\ -\frac{3t+2}{6t+1} & \frac{3t+2}{6t+1} \end{pmatrix}.$$

Thus

$$\mathbf{Y}_1(t) = \begin{pmatrix} \frac{3t-1}{6t+1} \\ -\frac{3t+2}{6t+1} \end{pmatrix}, \quad \mathbf{Y}_2(t) = \begin{pmatrix} \frac{3t+2}{6t+1} \\ \frac{3t+2}{6t+1} \end{pmatrix}.$$

By simple calculation we arrive at the following representation of the matrix $B(t)$

$$B(t) = B^+(t)D_3(t)B^-(t),$$

where

$$B^{+}(t)=\begin{pmatrix} 1 & \dfrac{1}{t-3} & 0 \\ 3 & \dfrac{1}{t-2} & 0 \\ 0 & 0 & 1 \end{pmatrix},\ B^{-}(t)=\begin{pmatrix} 1 & -1 & 0 \\ 1 & \dfrac{3t-1}{3t+2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$D_3(t)=\begin{pmatrix} t^2 & 0 & 0 \\ 0 & 1 & 0 \\ -\dfrac{3}{(t+2)(6t+1)} & \dfrac{2(3t+2)}{(t+2)(6t+1)} & \dfrac{t-2}{t+3}\dfrac{3t+2}{3t-1} \end{pmatrix}$$

Note that without loss of generality we can take element $c(t)$ of the matrix $D_3(t)$ equal to 1. Really, the function $c(t)$ possesses the following factorization

$$c(t)=\frac{t-2}{t+3}\cdot\frac{3t+2}{3t-1}=c^{+}(t)\cdot c^{-}(t).$$

Hence, by taking

$$\widetilde{B}^{+}(t)=\begin{pmatrix} 1 & \dfrac{1}{t-3} & 0 \\ 3 & \dfrac{1}{t-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c^{+}(t) \end{pmatrix},$$

$$\widetilde{B}^{+}(t)=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c^{-}(t) \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 1 & \dfrac{3t-1}{3t+2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we obtain the following representation of the matrix $B(t)$:

$$B(t)=\widetilde{B}^{+}(t)\widetilde{D}_3(t)\widetilde{B}^{-}(t),$$

with

$$\widetilde{D}_3(t)=\begin{pmatrix} t^2 & 0 & 0 \\ 0 & 1 & 0 \\ -\dfrac{3(t+3)}{(t+2)(6t+1)(t-2)} & \dfrac{2(3t+2)(t+3)}{(t+2)(6t+1)(t-2)} & 1 \end{pmatrix}.$$

It was shown in [8] that factorization problem for the matrix $G(t)$ is equivalent to the construction of the *canonical matrix-functions* $X^{\pm}(z)$, i.e. matrix-functions satisfying the homogeneous boundary condition

$$X^{+}(t)=G(t)X^{-}(t),\quad t\in\Gamma, \tag{4}$$

such that $X^-(z)$ has the normal form at infinity, i.e. the sum of the orders at infinity of the columns of $X^-(z)$ is equal to the index of the determinant of the matrix $G(t)$

$$\kappa = ind_\Gamma\, G(t) = wind_\Gamma\, G(t).$$

Note that the order of a column of the analytic matrix-function is equal to the minimal order of the elements of the column.

Therefore, instead of the direct determination of a solution to the factorization problem we can construct the canonical matrix-function for the matrix boundary value problem (4). Moreover, it follows from Lemma 2, that we can take a non-singular triangular matrix-function of the third order $G(t)$ in the special form, namely

$$G(t) = \begin{pmatrix} \zeta_1(t) & 0 & 0 \\ 0 & \zeta_2(t) & 0 \\ a_1(t) & a_2(t) & 1 \end{pmatrix} \tag{5}$$

As before, we suppose that all entries of the matrix are Hölder-continuous on Γ and indices of $\zeta_1(t), \zeta_2(t)$ are equal κ_1, κ_2, respectively. Note that by inductive consideration the same form can be taken for the matrices of higher order, i.e. with diagonal entries $(\zeta_1(t), \zeta_2(t), \ldots, \zeta_{n-1}(t), 1)$, the entries of the last row $(a_1(t), a_2(t), \ldots, a_{n-1}(t), 1)$, and remaining entries equal to zero.

Let us present few details of the algorithm proposed in [9] for the matrix-function of the form (5). First, the functions $\zeta_j(t), j = 1, 2$, satisfy the following factorization equality

$$x_j^+(t) = \zeta_j(t) x_j^-(t), \quad t \in \Gamma.$$

Introduce the functions

$$\phi_j^\pm(z) = \frac{1}{2\pi i} \int\limits_\Gamma \frac{a_j(\tau) x_j^-(\tau) d\tau}{\tau - z}, \quad z \in D^\pm, j = 1, 2.$$

Then the analytic in $D^\pm$ matrices

$$X^\pm(z) = \begin{pmatrix} x_1^\pm(t) & 0 & 0 \\ 0 & x_2^\pm(t) & 0 \\ \phi_1^\pm(t) & \phi_2^\pm(t) & 1 \end{pmatrix}$$

satisfy the boundary condition (4). Denote by $\gamma_1 \geq 1, \gamma_2 \geq 1$ the orders of the functions $\phi_1^-(z), \phi_2^-(z)$ at infinity.

If $\kappa_1 \le \gamma_1$, $\kappa_2 \le \gamma_2$, then $X^-(z)$ has the normal form at infinity and thus $X^\pm(z)$ is the canonical matrix. In this case partial indices are equal $(\kappa_1, \kappa_2, 0)$. If at least one of the above inequalities fails, then $X^-(z)$ does not have the normal form at the infinity. In this case it is necessary to do elementary transformations with the columns of $X^-(z)$ (see for details [9]).

Example 2. *Let us consider factorization problem for the matrix-function*

$$G(t) = \begin{pmatrix} t^2(t+2) & 0 & 0 \\ 0 & \frac{t+2}{t+3}\frac{2t-1}{2t+1} & 0 \\ -\frac{3(t+3)}{(t-2)(2t+1)(6t+1)} & \frac{2(3t+2)(t+3)}{(t-2)(2t+1)(6t+1)} & 1 \end{pmatrix}.$$

In this case $x_1^+(z) = z+2$, $x_1^-(z) = \frac{1}{z^2}$, and $x_2^+(z) = \frac{z+2}{z+3}$, $x_2^-(z) = \frac{2z+1}{2z-1}$, and indices of the diagonal elements are equal $(\kappa_1, \kappa_2, 0) = (2, 0, 0)$. Consider the functions

$$\phi_1(z) = -\frac{3(z+3)}{(z-2)(2z+1)(6z+1)} x_1^-(z) = -\frac{3(z+3)}{z^2(z-2)(2z+1)(6z+1)},$$

$$\phi_2(z) = \frac{2(3z+2)(z+3)}{(z-2)(2z+1)(6z+1)} x_2^-(z) = \frac{2(3z+2)(z+3)}{(z-2)(2z-1)(6z+1)}.$$

Let us expand the functions $\phi_1(t), \phi_2(t)$ in simple fractions

$$\phi_1(t) = \frac{-129/4}{t} + \frac{9/2}{t^2} + \frac{-6}{2t+1} + \frac{2754/13}{6t+1} + \frac{-3/52}{t-2},$$

$$\phi_2(t) = \frac{-49/12}{2t-1} + \frac{153/52}{6t+1} + \frac{80/39}{t-2}.$$

Thus

$$\phi_1^-(t) = \frac{-129/4}{t} + \frac{9/2}{t^2} + \frac{-6}{2t+1} + \frac{2754/13}{6t+1}, \quad \phi_2^-(t) = \frac{-49/12}{2t-1} + \frac{153/52}{6t+1}.$$

Hence $\gamma_1 = 1 < \kappa_1 = 2$, but $\gamma_2 = 1 > \kappa_2 = 0$. Therefore we have to do elementary transformations with the first and third columns. For this we expand the function $1/\phi_1^-(t)$ in continued fraction

$$\frac{1}{\phi_1^-(z)} = \frac{52(12z^4 + 8z^3 + z^2)}{3(12z^3 + 32z^2 + 65z + 78)} = q_1^{\gamma_{1,0}}(z) + \frac{1}{q_1^{\gamma_{1,1}}(z) + \frac{1}{q_1^{\gamma_{1,2}}(z) + \ldots}}.$$

Here $q_1^{\gamma_{1,0}}(z) = 52/3z - 104/3$ is a polynomial of the order 1. Applying Chebotarev's algorithm we get the partial Multiplying the first column on $-q_1^{\gamma_{1,0}}(z)$ and adding to the third column we obtain that the transformed matrix in the form

$$\widetilde{X}^-(z) = \begin{pmatrix} x_1^{\pm}(z) & 0 & -\dfrac{52/3}{z} + \dfrac{104/3}{z^2} \\ 0 & x_2^{\pm}(z) & 0 \\ \phi_1^{\pm}(z) & \phi_2^{\pm}(z) & \dfrac{52(z+3)}{z^2(2z+1)(6z+1)} \end{pmatrix}.$$

It has the normal form at infinity and its partial indices are equal (1,0,1)[1].

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[1]Calculation in these examples are performed by using “Alfa Mathematica”.

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Аннотация

Дубатовская М. В., Примачук Л. П., Рогозин С. В. О факторизации треугольных матриц функций

Статья посвящена анализу эффективного метода факторизации треугольных матриц функций произвольного порядка, обобщающего метод Г. Н.Чеботарева. Результаты проиллюстрированы примерами.

Ключевые слова: факторизация матриц-функций, треугольные матрицы, цепные дроби.

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