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THE GROWTH OF L^p-NORMS IN PRESENCE OF LOGARITHMIC SOBOLEV INEQUALITIES[1 2]

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The growth of L^p-norms is considered under various hypotheses, including LS_q (logarithmic Sobolev) inequalities.

1. The graph setting

Let $G = (V, \mathcal{E})$ be a connected, non-oriented, finite graph. It will be equipped with the uniform probability measure μ, assigning the mass $1/\mathrm{card}(V)$ to each point of V, where $\mathrm{card}(V)$ is the number of vertices.

If $(x, y) \in \mathcal{E}$, we will say that the points x and y are neighbours and write $y \sim x$. With any real-valued function f on V, associate the modulus of the gradient $|\nabla f| \geq 0$, defined on V by

$$|\nabla f(x)|^2 = \sum_{y \sim x} |f(x) - f(y)|^2.$$

Various relations between the distributions of f and $|\nabla f|$ under the measure μ are the subject of the theory of Sobolev inequalities on finite graphs, which may be included as part in the framework of finite Markov chains, cf. e.g. [SC], [L2], [B-T], [G-M-T], [M-T]. Of the most interest are Poincaré-type and logarithmic Sobolev inequalities

$$\lambda_1 \mathrm{Var}(f) \leq \int |\nabla f|^2 \, d\mu, \tag{1.1}$$

$$\rho \, \mathrm{Ent}(f^2) \leq 2 \int |\nabla f|^2 \, d\mu, \tag{1.2}$$

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which are supposed to hold for all $f : V \to \mathbf{R}$. Here, $\mathrm{Var}(f) = \int f^2\,d\mu - (\int f\,d\mu)^2$ stands for the variance of f, and

$$\mathrm{Ent}(f^2) = \int f^2 \log f^2\,d\mu - \int f^2\,d\mu\,\log \int f^2\,d\mu$$

for the entropy of f^2 under μ. An optimal constant $\lambda_1 > 0$ is called the spectral gap, while an optimal value $\rho > 0$ is referred to as the logarithmic Sobolev constant of G. It is well-known that $\rho \leq \lambda_1$, which immediately follows from the relation

$$\lim_{C\to\infty} \mathrm{Ent}\left((f+C)^2\right) = 2\,\mathrm{Var}(f).$$

One of the interesting problems is how to relate the L^p-norms $\|f\|_p = (\int |f|^p\,d\mu)^{1/p}$ of the functions on V to the L^p-norms of their moduli of the gradients

$$\|\nabla f\|_p = \left(\int |\nabla f|^p\,d\mu\right)^{1/p}.$$

Usually, one considers a more specific problem about the probability

$$\mu\{|f - \mathbf{E}f| \geq r\}, \quad r > 0,$$

of large deviations of f from the mean $\mathbf{E}f = \int f\,d\mu$ in the class of Lipschitz functions. See e.g. [A-M], [L1-2], and references therein. A closely related question is the one about the rate of the growth of L^p-norms ([R2]). Note that in the graph setting the Lipschitz property may be understood in many senses. As a natural choice, the value $\|f\|_{\mathrm{Lip}} = \max_{x\in V} |\nabla f(x)|$ is called the Lipschitz constant of f, and one says that f is Lipschitz, if $\|f\|_{\mathrm{Lip}} \leq 1$.

However, often the Lipschitz constant does not reflect many properties of the distributions of functions, and it is desirable to have more sensitive bounds for large deviations. For example, in presence of the logarithmic Sobolev inequality (1.2), one always has

$$\int e^f\,d\mu \leq \int e^{|\nabla f|^2/\rho}\,d\mu \tag{1.3}$$

for any f on V with μ-mean zero ([B-G]). This can also be formulated in terms of the Orlicz norms $\|\cdot\|_{\psi_\alpha}$, generated by the Young functions $\psi_\alpha(t) = e^{|t|^\alpha} - 1$ with $\alpha = 1$ and $\alpha = 2$. Let us recall that

$$\|f\|_{\psi_\alpha} = \inf\left\{t > 0 : \int \psi_\alpha(f/t)\,d\mu \leq 1\right\}.$$

Put also $\|\nabla f\|_{\psi_\alpha} = \|\,|\nabla f|\,\|_{\psi_\alpha}$. Then, by the very definition, (1.3) yields the relation

$$\|f\|_{\psi_1} \leq \frac{2}{\sqrt{\rho}}\,\|\nabla f\|_{\psi_2}. \tag{1.4}$$

It would be interesting to study whether the converse implication is true, as well. One may also wonder how to refine the inequality (1.4) in terms of L^p-norms. Indeed, as is known and easy to see (see Appendix), up to some absolute constants $c_1, c_2 > 0$,

$$c_1 \sup_{p \geq 1} \frac{\|f\|_p}{p} \leq \|f\|_{\psi_1} \leq c_2 \sup_{p \geq 1} \frac{\|f\|_p}{p}, \tag{1.5}$$

while

$$c_1 \sup_{p \geq 1} \frac{\|f\|_p}{\sqrt{p}} \leq \|f\|_{\psi_2} \leq c_2 \sup_{p \geq 1} \frac{\|f\|_p}{\sqrt{p}}. \tag{1.6}$$

Hence, as (1.4) suggests, one may expect that $\|f\|_p$ may be bounded by $\|\nabla f\|_p$ with factors growing like $\sqrt{p}$. For the first time, an observation of this kind was apparently made in the 1994 work by S. Aida and D. Stroock [A-S] for an abstract scheme of Markov transition kernels. They also considered defective logarithmic Sobolev inequalities

$$\rho\,\mathrm{Ent}(f^2) \leq 2\,(\mathrm{E}f, f) + \beta \int |\nabla f|^2\,d\mu,$$

where $(\mathrm{E}f, f)$ is a given Dirichlet form, and where an additional term $\beta \int |\nabla f|^2\,d\mu$ is referred to as defect.

The aim of this note is to adapt the approach of [A-S] to the graph setting as above and to the setting of abstract metric spaces with local moduli of the gradients. We will also see that more general hypotheses in comparison with (1.2), such as, the so-called LS_q-inequalities may be involved in a similar analysis.

2. The growth of L^p-norms on finite graphs

Keeping the same notations as in Section 1, the following theorem holds.

Theorem 2.1. *For any function f on V and any $p > 2$,*

$$\|f\|_p^2 - \|f\|_2^2 \leq \frac{2}{\rho} \int_2^p \|\nabla f\|_t^2\,dt, \tag{2.1}$$

where ρ is the logarithmic Sobolev constant. In particular,

$$\|f\|_p^2 - \|f\|_2^2 \leq \frac{2(p-2)}{\rho}\,\|\nabla f\|_p^2. \tag{2.2}$$

Essentially, this statement is a variation of Theorem 3.4 in [A-S] (in the case where there is no defect).

Note that, dividing the inequality (2.2) by $p-2$ and letting $p \to 2$, one arrives at the logarithmic Sobolev inequality (1.2) with an additional factor 2 on the right-hand side. In this sense, (2.2) and (1.2) are almost equivalent.

Under an additional mean assumption $\mathbf{E}f = 0$, the term $\|f\|_2$ in (2.2) may further be estimated by using the Poincaré-type inequality (1.1). More precisely, it gives

$$\|f\|_2^2 \leq \frac{1}{\lambda_1}\,\|\nabla f\|_2^2 \leq \frac{1}{\rho}\,\|\nabla f\|_p^2.$$

Hence, we also have:

Corollary 2.2. *For any function f on V with μ-mean zero,*

$$\|f\|_p \ \leq \ \sqrt{\frac{2p-3}{\rho}}\ \|\nabla f\|_p. \tag{2.3}$$

Let us remind the argument of [A-S], which goes back to the seminal work of L. Gross [G]. It is based on the general formula

$$\frac{d}{dp}\,\|f\|_p \ = \ \frac{1}{p^2}\,\|f\|_p^{1-p}\,\mathrm{Ent}(|f|^p), \quad p > 0, \tag{2.4}$$

holding for any measurable function f on an arbitrary probability space, such that $0 < \|f\|_{p+\varepsilon} < +\infty$, for some $\varepsilon > 0$. In the graph setting the latter just means that f is not identically zero. Therefore, for all $p > 0$,

$$\frac{d}{dp}\,\|f\|_p^2 \ = \ \frac{2}{p^2}\,\|f\|_p^{2-p}\,\mathrm{Ent}(|f|^p). \tag{2.5}$$

Let $p > 2$ and $\|f\|_p > 0$. According to the hypothesis (1.2), applied to the function $|f|^{p/2}$, we may write

$$\rho\,\mathrm{Ent}(|f|^p) \leq 2\int |\nabla |f|^{p/2}|^2\,d\mu. \tag{2.6}$$

Here the right-hand side is

$$2\int \sum_{y\sim x} \big(|f(x)|^{p/2} - |f(y)|^{p/2}\big)^2\,d\mu(x) \ = \ \frac{4}{\mathrm{card}(V)}\,\sum_x\sum_{y\sim x} \big(|f(x)|^{p/2} - |f(y)|^{p/2}\big)^2\,1_{\{|f(x)|>|f(y)|\}}.$$

Since $a^r - b^r \leq ra^{r-1}(a-b)$, for all $a \geq b \geq 0$ and $r \geq 1$, the expression inside the double sum is estimated by

$$\frac{p^2}{4}\,(|f(x)| - |f(y)|)^2\,|f(x)|^{p-2}\,1_{\{|f(x)|>|f(y)|\}} \;\leq\; \frac{p^2}{4}\,(f(x)-f(y))^2\,|f(x)|^{p-2}.$$

Thus,

$$\begin{aligned} 2\int |\nabla |f|^{p/2}|^2\,d\mu \;&\leq\; \frac{4}{\mathrm{card}(V)}\sum_x\sum_{y\sim x}\frac{p^2}{4}\,(f(x)-f(y))^2\,|f(x)|^{p-2} \\ &=\; p^2\int |\nabla f(x)|^2\,|f(x)|^{p-2}\,d\mu(x), \end{aligned}$$

and by (2.6),

$$\mathrm{Ent}(|f|^p) \;\leq\; \frac{p^2}{\rho}\int |\nabla f|^2\,|f|^{p-2}\,d\mu.$$

The last integral may further be bounded by virtue of Hölder's inequality with the exponents $\frac{p}{2}$ and $(\frac{p}{2})^* = \frac{p}{p-2}$, and we get

$$\mathrm{Ent}(|f|^p) \;\leq\; \frac{p^2}{\rho}\,\|\nabla f\|_p^2\,\|f\|_p^{p-2}.$$

Hence from (2.5) we obtain a differential inequality $\frac{d}{dp}\,\|f\|_p^2 \;\leq\; \frac{2}{\rho}\,\|\nabla f\|_p^2$, which can easily be integrated to yield (2.1).

3. LS_q-inequalities

In the literature one can find a variety of more general or modified forms of the logarithmic Sobolev inequality (1.2), which have been introduced and analysed for different aims; cf. e.g. [L2-3], [B-T]. Here we consider one natural generalization, which in the graph setting as above takes the form

$$\mathrm{Ent}(|f|^q) \leq C_q^q \int |\nabla f(x)|_q^q\,d\mu(x), \tag{3.1}$$

where $1 \leq q \leq 2$ is a fixed parameter, C_q is a positive constant (independent of the function f on V), and where

$$|\nabla f(x)|_q \;=\; \Big(\sum_{y\sim x}|f(x)-f(y)|^q\Big)^{1/q}, \quad x \in V. \tag{3.2}$$

Definition. The inequality (3.1) will be called a logarithmic Sobolev inequality with the power parameter q, or for short, an LS_q-inequality (with constant C_q).

Under the same name a similar family of analytic inequalities for local gradients was treated in [B-Z] to study super-Gaussian tails for distributions of Lipschitz functions. As it turns out, (3.1) may also be used to obtain improved rates for the growth of L^p-norms in comparison with the particular case $q = 2$ of Theorem 2.1.

Indeed, given $p > q$, and starting with the identity (2.4), we get

$$\frac{d}{dp}\,\|f\|_p^q \;=\; \frac{q}{p^2}\,\|f\|_p^{q-p}\,\mathrm{Ent}(|f|^p), \tag{3.3}$$

where one should assume that $\|f\|_p > 0$. On the other hand, by (3.1), applied to the function $|f|^{p/q}$, we have

$$\begin{aligned}
\mathrm{Ent}(|f|^p) \;&\le\; C_q^q \int |\,\nabla |f|^{p/q}\,|_q^q\, d\mu \\
&=\; C_q^q \int \sum_{y\sim x} \left(|f(x)|^{p/q} - |f(y)|^{p/q}\right)^q\, d\mu(x) \\
&=\; \frac{2C_q^q}{\mathrm{card}(V)} \sum_x \sum_{y\sim x} \left(|f(x)|^{p/q} - |f(y)|^{p/q}\right)^q\, 1_{\{|f(x)|>|f(y)|\}}.
\end{aligned}$$

The expression inside the double sum can be bounded by

$$\left(\frac{p}{q}\right)^q (|f(x)|-|f(y)|)^q\,|f(x)|^{p-q}\,1_{\{|f(x)|>|f(y)|\}} \;\le\; \left(\frac{p}{q}\right)^q |f(x)-f(y)|^q\,|f(x)|^{p-q}.$$

Hence,

$$\begin{aligned}
\mathrm{Ent}(|f|^p) \;&\le\; \frac{2C_q^q}{\mathrm{card}(V)} \left(\frac{p}{q}\right)^q \sum_x \sum_{y\sim x} |f(x) - f(y)|^q\,|f(x)|^{p-q} \\
&=\; \frac{2p^q\,C_q^q}{q^q} \int |\nabla f(x)|_q^q\,|f(x)|^{p-q}\, d\mu(x).
\end{aligned}$$

By Hölder's inequality with the exponents $\frac{p}{q}$ and $(\frac{p}{q})^* = \frac{p}{p-q}$, the above integral is estimated by

$$\left(\int |\nabla f(x)|_q^p\, d\mu(x)\right)^{q/p} \left(\int |f(x)|^p\, d\mu(x)\right)^{(p-q)/p} \;=\; \|\,|\nabla f|_q\,\|_p^q\;\|f\|_p^{p-q},$$

and we get

$$\mathrm{Ent}(|f|^q) \;\le\; \frac{2p^q\,C_q^q}{q^q}\;\|\,|\nabla f|_q\,\|_p^q\;\|f\|_p^{p-q}.$$

Combining this with (3.3), we obtain a differential inequality

$$\frac{d}{dp}\,\|f\|_p^q \ \le\ 2C_q^q\,\frac{p^{q-2}}{q^{q-1}}\,\|\,|\nabla f|_q\,\|_p^q. \tag{3.4}$$

If $q > 1$, integrate (3.4) between q and p with respect to the variable p, and then we arrive at

$$\|f\|_p^q - \|f\|_q^q \ \le\ \frac{2C_q^q}{q^{q-1}}\int_q^p t^{q-2}\,\|\,|\nabla f|_q\,\|_t^q\,dt \ \le\ \frac{2C_q^q}{q^{q-1}}\,\frac{p^{q-1}-q^{q-1}}{q-1}\,\|\,|\nabla f|_q\,\|_p^q.$$

Thus, we have obtained the following:

Theorem 3.1. *Whenever $p > q > 1$, under an LS_q-inequality with constant C_q, we have, for any function f on V,*

$$\|f\|_p^q - \|f\|_q^q \ \le\ 2C_q^q\,\frac{(\frac{p}{q})^{q-1}-1}{q-1}\,\|\,|\nabla f|_q\,\|_p^q. \tag{3.5}$$

4. LS_1-inequality and isoperimetry

The limit case $q = 1$ in Theorem 3.1 is special, since then (3.4) turns into the differential inequality

$$\frac{d}{dp}\,\|f\|_p \ \le\ \frac{2C_1}{p}\,\|\,|\nabla f|_1\,\|_p.$$

Clearly, it yields

$$\|f\|_p - \|f\|_1 \ \le\ 2C_1\int_q^p \frac{1}{t}\,\|\,|\nabla f|_1\,\|_t\,dt \ \le\ 2C_1\,\log p\,\|\,|\nabla f|_1\,\|_p,$$

which may also be obtained by letting $q \to 1$ in (3.5).

Hence, if an LS_1-inequality is satisfied for the graph $G = (V, \mathcal{E})$ with constant C_1, that is, if for any function f on V,

$$\mathrm{Ent}(|f|) \le C_1 \int \sum_{y\sim x} |f(x) - f(y)|\,d\mu(x), \tag{4.1}$$

then, for any $p > 1$, we have

$$\|f\|_p - \|f\|_1 \ \le\ 2C_1\,\log p\,\bigg(\int\Big(\sum_{y\sim x}|f(x) - f(y)|\Big)^p\,d\mu(x)\bigg)^{1/p}. \tag{4.2}$$

Thus, while in case $q > 1$ the coeffcient in front of $\| |\nabla f|_q \|_p$ grows polynomially fast as in Theorem 3.1, it grows logarithmically in case $q = 1$.

Now, it might be useful to realize that the hypothesis (4.1) has a simple geometric description of an isoperimetric nature. The main feature of this inequality is that it holds for all f, if and only if it holds in the class of the indicator functions $f = 1_A$, $A \subset V$.

To see this, first note that, by homogeneity and due to the triangle inequality $|\,|a| - |b|\,| \leq |a - b|$ $(a, b \in \mathbf{R})$, without loss of generality one may restrict (4.1) to the functions with values $0 \leq f \leq 1$. Let us assume this. Introduce a measure Q on $V \times V$, assigning the mass $1/\mathrm{card}(V)$ to each couple (x, y) of neighbours in V. Put $A_t = \{x \in V : f(x) \geq t\}$, so that

$$f(x) = \int_0^1 1_{A_t}(x)\, dt \tag{4.3}$$

and

$$\int \sum_{y \sim x} |f(x) - f(y)|\, d\mu(x) = 2 \int_0^1 Q(A_t \times \bar{A}_t)\, dt,$$

where $\bar{A}_t$ denotes the complement of A_t in V. Then, (4.1) takes the form

$$\mathrm{Ent}(f) \leq 2C_1 \int_0^1 Q(A_t \times \bar{A}_t)\, dt. \tag{4.4}$$

But the entropy $f \to \mathrm{Ent}(f)$ is a convex functional, so, by (4.3),

$$\mathrm{Ent}(f) \leq \int_0^1 \mathrm{Ent}(1_{A_t})\, dt.$$

Hence, (4.4) would follow from

$$\mathrm{Ent}(1_{A_t}) \leq 2C_1\, Q(A_t \times \bar{A}_t), \quad 0 < t < 1.$$

It remains to note that the inequality of the form

$$\mathrm{Ent}(1_A) \leq 2C_1\, Q(A \times \bar{A}) \tag{4.5}$$

is exactly (4.1) for $f = 1_A$.

Now, with every set A in V we associate the edge boundary function

$$h_A(x) = \text{the number of neighbours of } x \text{ outside } A, \qquad x \in A,$$

putting $h_A(x) = 0$, if $x \notin A$. Then clearly $Q(A \times \bar{A}) = \int h_A\, d\mu$.

In addition, $\mathrm{Ent}(1_A) = \mu(A) \log \frac{1}{\mu(A)}$. Replacing $C = 2C_1$ in (4.2) and (4.5), we arrive at:

Theorem 4.1. *Assume that, for any set $A \subset V$,*

$$\mu(A) \log \frac{1}{\mu(A)} \leq C \int h_A(x)\, d\mu(x) \tag{4.6}$$

with some constant C. Then, for any function f on V, and any $p > 1$,

$$\|f\|_p - \|f\|_1 \ \leq \ C \log p \left(\int \Big(\sum_{y \sim x} |f(x) - f(y)| \Big)^p \, d\mu(x) \right)^{1/p}.$$

The inequality (4.6) is of an isoperimetric type: the integral on the right-hand side represents the size of the edge boundary of A.

5. Connection between LS_q and Poincaré-type inequalities

In order to judge about the rate of the growth of the functions f on V with mean $\mathbf{E}f = \int f\, d\mu = 0$ in Theorem 3.1, so that to obtain a corresponding analogue of Corollary 2.2, we need to relate LS_q-inequalities

$$\mathrm{Ent}(|f|^q) \ \leq \ C_q^q \int \sum_{y \sim x} |f(x) - f(y)|^q \, d\mu(x), \tag{5.1}$$

to the inequalities of Poincaré-type

$$\|f - \mathbf{E}f\|_q^q \leq A_q^q \int \sum_{y \sim x} |f(x) - f(y)|^q \, d\mu(x). \tag{5.2}$$

Here we assume that C_q and A_q are some positive constants independent of f.

Note that (5.2) makes sense for any $q \geq 1$, while (5.1) requires that $q \leq 2$. Indeed, otherwise, if f is not identically zero, $\mathrm{Ent}(|f + C|^q) \to +\infty$, as $C \to +\infty$, while the right-hand side of (5.1) does not change. So, (5.1) may not hold for $q > 2$.

As was already mentioned, in the basic case $q = 2$, (5.1) easily implies (5.2) with $A_2^2 = \frac{1}{2} C_2^2$, by applying the logarithmic Sobolev inequality to the functions of the form $f + C$ with growing C. This argument is no longer valid in case $1 \leq q < 2$, since then $\mathrm{Ent}(|f + C|^q) \to 0$, as $C \to +\infty$. Therefore, we need to choose a different route.

Lemma 5.1. *For any function $f \geq 0$ on V,*

$$\mathrm{Ent}(f) \geq -\log \mu\{f > 0\}\, \mathbf{E}f. \tag{5.3}$$

This observation is elementary, and we refer to [B-Z], Lemma 2.2.

Now, to derive (5.2) from (5.1), take an arbitrary function f on V with μ-median at zero, that is, such that $\mu\{f > 0\} \leq \frac{1}{2}$ and $\mu\{f < 0\} \leq \frac{1}{2}$. Apply (5.3) to the functions $(f^+)^q$ and $(f^-)^q$, where

$$f^+ = \max\{f, 0\}, \quad f^- = \max\{-f, 0\}.$$

Then we get

$$\mathrm{Ent}((f^+)^q) \,\geq\, \log 2\, \mathbf{E}\,(f^+)^q, \qquad \mathrm{Ent}((f^-)^q) \,\geq\, \log 2\, \mathbf{E}\,(f^-)^q,$$

and after summing

$$\mathbf{E}\,|f|^q \,\leq\, \frac{1}{\log 2}\,\big(\mathrm{Ent}((f^+)^q) + \mathrm{Ent}((f^-)^q)\big). \tag{5.4}$$

On the other hand, $|f^+(x) - f^+(y)| \leq |f(x) - f(y)|$, and similarly for f^-. Hence, combining (5.1) with (5.4), we obtain that

$$\mathbf{E}\,|f|^q \,\leq\, \frac{2C_q^q}{\log 2} \int \sum_{y \sim x} |f(x) - f(y)|^q \, d\mu(x). \tag{5.5}$$

Finally, $\|f - \mathbf{E}f\|_q \leq \|f\|_q + \mathbf{E}\,|f| \leq 2\|f\|_q$, so $\mathbf{E}\,|f - \mathbf{E}f|^q \leq 2^q \mathbf{E}\,|f|^q$, and by (5.5),

$$\mathbf{E}\,|f - \mathbf{E}f|^q \,\leq\, \frac{2^{1+q}\, C_q^q}{\log 2} \int \sum_{y \sim x} |f(x) - f(y)|^q \, d\mu(x). \tag{5.6}$$

The both sides of this inequality are invariant under adding any constant to f, so the assumption about the median of f may be removed at this step. Hence, (5.6) provides a Poincaré-type inequality such as (5.2).

Theorem 5.2. *Under the* LS_q*-inequality* (5.1) *with constant* C_q*, the Poincaré-type inequality* (5.2) *holds true with*

$$A_q^q = \frac{2^{1+q}\, C_q^q}{\log 2}.$$

We can now return to Theorem 3.1. Under the mean assumption $\mathbf{E}f = 0$, using Theorem 5.2, we get that

$$\begin{aligned}\|f\|_p^q &\le \|f\|_q^q + 2C_q^q \, \frac{(\frac{p}{q})^{q-1} - 1}{q-1} \, \|\,|\nabla f|_q\,\|_p^q \\ &\le 2\left(\frac{2^q}{\log 2} + \frac{(\frac{p}{q})^{q-1} - 1}{q-1}\right) C_q^q \, \|\,|\nabla f|_q\,\|_p^q.\end{aligned}$$

To simplify the constant, assume $p \ge 2$. Clearly, $(\frac{p}{q})^{q-1} \le p^{q-1}$, and

$$\frac{p^{q-1} - 1}{q-1} \ge \log p \ge \log 2 \ge c\,\frac{2^q}{\log 2}$$

with $c = \frac{\log^2 2}{4}$. Hence, $\|f\|_p^q \le 2\big(1 + \frac{1}{c}\big)\,\frac{p^{q-1}-1}{q-1}\,C_q^q \, \|\,|\nabla f|_q\,\|_p^q$.

Up to an absolute factor, this is a generalized form of Corollary 2.2.

Corollary 5.3. *Under the* LS_q*-inequality* (5.1) *with constant* C_q $(1 \le q \le 2)$, *for any function f on V with μ-mean zero, and any $p \ge 2$,*

$$\|f\|_p \le 20\, C_q \left(\frac{p^{q-1} - 1}{q-1}\right)^{1/q} \|\,|\nabla f|_q\,\|_p.$$

6. Continuous setting

The above statements remain to hold for local gradients. To describe a general scheme, let (V, d) be a metric space, equipped with a Borel probability measure μ. Given a function f on V, define the generalized modulus of the gradient

$$|Df(x)| = \limsup_{d(x,y)\to 0^+} \frac{|f(x) - f(y)|}{d(x,y)}, \quad x \in V, \tag{6.1}$$

putting $|Df(x)| = 0$, if x is an isolated point in V.

If f is continuous, then $|Df|$ is Borel measurable (cf. [B-H]).

A function f is called locally Lipschitz, if for any point $x \in V$, there is $r > 0$ and $C = C(x,r)$, such that, $|f(x) - f(y)| \le Cd(x,y)$, whenever $d(x,y) < r$ (that is, f has a finite Lipschitz constant on some neighborhood of x). For such functions, $|Df|$ are finite and Borel measurable.

For example, if V is the Euclidean space $\mathbf{R}^n$ with Euclidean metric, any locally Lipschitz function f on V is differentiable almost everywhere (with respect to Lebesgue measure), and (6.1) leads to

$$|Df(x)|^2 = \sum_{i=1}^{n} \left| \frac{\partial f(x)}{\partial x_i} \right|^2$$

at every point x, at which f is differentiable.

Returning to the abstract metric probability space, we say that (V, d, μ) satisfies an LS_q-inequality with constant C_q, where $1 \le q \le 2$, if for any bounded locally Lipschitz function f on V,

$$\mathrm{Ent}(|f|^q) \le C_q^q \int |Df(x)|^q \, d\mu(x). \tag{6.2}$$

The requirement that f is bounded may easily be removed from the definition.

In the sequel, for short, we write $\|Df\|_p = \big(\int |Df|^p \, d\mu\big)^{1/p}$.

The corresponding analogue of Theorem 3.1 is the following.

Theorem 6.1. *Whenever $p > q > 1$, under an LS_q-inequality with constant C_q, for any locally Lipschitz function f on V with finite $\|f\|_p$,*

$$\|f\|_p^q - \|f\|_q^q \ \le \ C_q^q \, \frac{(\frac{p}{q})^{q-1} - 1}{q-1} \, \|Df\|_p^q. \tag{6.3}$$

The proof is similar to the proof of Theorem 3.1 and is even simpler due to the property $|D\,u(f)| \le |u'(f)| \cdot |D(f)|$, where u is an arbitrary differentiable function on the real line. (As a result, the factor 2 in (3.5) can be removed).

In the limit case $q = 1$, the LS_1-inequality (6.2) is equivalent to the isoperimetric-type inequality

$$\mu(A) \log \frac{1}{\mu(A)} \le C_1 \mu^+(A) \tag{6.4}$$

in the class of all Borel subsets A of V, where

$$\mu^+(A) = \liminf_{r \to 0^+} \frac{\mu(A^r) - \mu(A)}{r}$$

is the outer Minkowski content of A with respect to the measure μ (A^r is the open r-neighbourhood of A for the metric d). For the entropy and related functionals, this equivalence was studied by many authors, starting with the works by V.G. Maz'ya in the early 1960s, cf. [M], [R1], [B-H].

As a result, we obtain:

Theorem 6.2. *In presence of the isoperimetric inequality* (6.4), *for any* $p > 1$ *and for any locally Lipschitz function* f *on* V *with finite* $\|f\|_p$,

$$\|f\|_p - \|f\|_1 \ \leq \ C_1 \log p \ \|Df\|_p.$$

Example. As shown in [B2], any log-concave probability measure μ, supported on a Euclidean ball in $\mathbf{R}^n$ of radius $r > 0$, satisfies the isoperimetric inequality

$$2r\, \mu^+(A) \geq \mu(A) \log \frac{1}{\mu(A)} + (1 - \mu(A)) \log \frac{1}{1 - \mu(A)}.$$

Hence, (6.4) is fulfilled with $C_1 = 2r$, so, by Theorem 6.2,

$$\|f\|_p - \|f\|_1 \ \leq \ 2r \log p \ \|Df\|_p \,, \qquad p > 1.$$

In particular, this holds for any convex body K in $\mathbf{R}^n$, contained in a Euclidean ball of radius $r > 0$, with respect to the normalized Lebesgue measure.

Let us return to Theorem 6.1. As shown in [B-Z], an LS_q-inequality (6.2) implies a Poincaré-type inequality

$$\|f\|_q^q \ \leq \ \frac{4\, C_q^q}{\log 2} \ \|Df\|_q^q \tag{6.5}$$

in the class of all locally Lipschitz functions f on V with μ-mean $\mathbf{E}f = 0$. Combining it with (6.3), we get, for any $p > q$,

$$\|f\|_p^q \ \leq \ \left(\frac{4}{\log 2} + \frac{(\frac{p}{q})^{q-1} - 1}{q - 1} \right) C_q^q \ \|Df\|_q^q.$$

One can simplify the constant like in the proof of Corollary 5.3 and derive:

Corollary 6.3. *Under the* LS_q*-inequality* (6.2) *with constant* C_q ($1 \leq q \leq 2$), *for any locally Lipschitz function* f *on* V *with* μ*-mean zero, and any* $p \geq 2$,

$$\|f\|_p \ \leq \ 10\, C_q \left(\frac{p^{q-1} - 1}{q - 1} \right)^{1/q} \|Df\|_p. \tag{6.6}$$

Examples. A probability measure μ on $\mathbf{R}$ with density $\frac{d\mu(x)}{dx} = c_1 \exp\{-c_2|x|^\alpha\}$ satisfies an LS$_q$-inequality (6.2) with finite constant, if and only if $\alpha \geq \frac{q}{q-1}$ $(1 < q \leq 2)$.

More generally, one may consider densities of the form $e^{-U(x)}$, where the function $U \in C^2(\mathbf{R})$ is supposed to increase near $+\infty$, to decrease near $-\infty$, and to satisfy $U''(x) = o(U'(x)^2)$, as $|x| \to +\infty$. Then (cf. [B-Z]), μ satisfies (6.2) with a finite C_q $(1 < q \leq 2)$, if and only if

$$\limsup_{|x|\to+\infty} \frac{U(x) + \log|U'(x)|}{|U'(x)|^q} < +\infty.$$

In the classical case $q = 2$, replacing $C_2^2 = 2/\rho$, the LS$_q$-inequality (6.2) takes the form

$$\rho \operatorname{Ent}(f^2) \leq 2 \int |Df(x)|^2 \, d\mu(x), \tag{6.7}$$

while the conclusion (6.3) of Theorem 6.1 reads as

$$\|f\|_p^2 - \|f\|_2^2 \leq \frac{p-2}{\rho} \|Df\|_p^2, \qquad p > 2. \tag{6.8}$$

It is slightly better then the bound of Theorem 2.1 about the graph setting. Moreover, (6.7) may be obtained from (6.8) by letting $p \to 2$, so these inequalities are equivalent.

As was already mentioned, (6.8) implies a Poincaré-type inequality $\rho\|f\|_2^2 \leq \|Df\|_2^2$ (given that $\mathbf{E}f = 0$), which is a little better than what follows from the bound (6.5) in case $q = 2$. Hence, one can also improve the constant in Corollary 6.4. Namely, we get

$$\|f - \mathbf{E}f\|_p \leq \sqrt{\frac{p-1}{\rho}} \, \|Df\|_p, \qquad p > 2. \tag{6.9}$$

For example, the standard Gaussian measure μ on $\mathbf{R}^n$ satisfies (6.7) with $\rho = 1$ ([G]), so, by (6.9), for any μ-integrable locally Lipschitz function f on $\mathbf{R}^n$,

$$\|f - \mathbf{E}f\|_p \leq \sqrt{p-1} \, \|Df\|_p, \qquad p > 2.$$

This inequality (with asymptotically equivalent constants) is well-known and can be obtained by different arguments (cf. e.g. [P]).

Remark 6.4. It is shown in [B-Z], cf. Theorem 6.1, that in presence of the the LS$_q$-inequality (6.2), for any locally Lipschitz function f on V with μ-mean zero,

$$\int e^{f(x)/B_q} \, d\mu(x) \leq \int e^{|Df(x)|^q} \, d\mu(x), \tag{6.10}$$

where $B_q^q = q^* q^{-q} C_q^q$ and $q^* = \frac{q}{q-1}$ is conjugate to $q \in (1,2]$. This is a more general form of (1.3), which corresponds to the classical case $q = 2$ in (6.10).

If the Orlicz norm $\|Df\|_{\psi_q} \leq 1$, the second integral in (6.10) does not exceed 2, so

$$\int e^{|f|/B_q}\, d\mu \leq \int e^{f/B_q}\, d\mu + \int e^{-f/B_q}\, d\mu \leq 4.$$

Hence, $\int e^{|f|/(2B_q)}\, d\mu \leq 2$, or, equivalently, $\|f\|_{\psi_1} \leq 2B_q$. Using $B_q \leq (q-1)^{-1/q}\, C_q$, we may conclude that

$$\|f\|_{\psi_1} \leq \frac{2C_q}{(q-1)^{1/q}}\, \|Df\|_{\psi_q}, \tag{6.11}$$

for any f on V with μ-mean zero (which is a corresponding generalization of (1.4), up to a universal factor).

However, the dependence of the constant in (6.11) on q, when it is close to 1, is asymptotically incorrect. A sharper inequality can be derived from Corollary (6.3), by using an L^p-description of the the Orlicz norm given in Proposition 8.1 below.

Indeed, assuming that $\|Df\|_{\psi_q} \leq 1$ and applying a lower bound in (8.1) with $\alpha = q$, we get that $\|f\|_p \leq 2p^{1/q}$, for all $p \geq 1$. Hence, by (6.6), for all $p \geq 2$,

$$\|f\|_p \ \leq\ 20\, C_q \bigg(\frac{p^q - p}{q-1}\bigg)^{1/q}.$$

An optimal constant in $(p^q - p)^{1/q} \leq Ap$ for the range $p \geq 2$ is attained for $p = 2$, so $A = (1 - 2^{1-q})^{1/q}$, and

$$\|f\|_p \ \leq\ 20\, C_q \bigg(\frac{1 - 2^{1-q}}{q-1}\bigg)^{1/q} p \ \leq\ (20 \log 2)\ C_q\, p.$$

For the range $1 \leq p < 2$, one may just use $\|f\|_p \leq \|f\|_2 \leq\ (40 \log 2)\ C_q p$.

Now, applying a lower bound in (8.1) with $\alpha = 1$, we get that $\|f\|_{\psi_1} \leq (80e \log 2)\ C_q$.

This may be summarized in the following.

Corollary 6.5. *Under the* LS_q*-inequality* (6.2) *with constant* C_q*, for any locally Lipschitz function* f *on* V *with* μ*-mean zero,*

$$\|f\|_{\psi_1} \leq 160\, C_q\, \|Df\|_{\psi_q}. \tag{6.12}$$

7. Discrete cube

As an example of a graph, consider the discrete cube $V = \{0,1\}^n$ with the uniform probability measure μ, assigning the mass 2^{-n} to each point. Each point $x = (x_1, \dots, x_n)$ in V has exactly n-neighbours $s_i(x)$, $i = 1, \dots, n$, with coordinates

$$\big(s_i(x)\big)_j = x_j \ (j \neq i), \qquad \big(s_i(x)\big)_i = 1 - x_i.$$

Hence, any function f on V has the associated modulus of the discrete gradient, given by

$$|\nabla f(x)|^2 = \sum_{i=1}^n |f(x) - f(s_i(x))|^2.$$

According to the Bonami-Gross theorem [Bon], [G], in this case $\lambda_1 = \rho = 4$, that is, for any f on the discrete cube, we have

$$\mathrm{Ent}(f^2) \leq \frac{1}{2} \int |\nabla f|^2 \, d\mu. \tag{7.1}$$

Therefore, applying Theorem 2.1 and Corollary 2.2, we obtain the following:

Corollary 7.1. *For any function $f : \{0,1\}^n \to \mathbf{R}$, and any $p \geq 2$,*

$$\|f\|_p^2 - \|f\|_2^2 \leq \frac{p-2}{2} \, \|\nabla f\|_p^2. \tag{7.2}$$

In particular, if f has μ-mean zero,

$$\|f\|_p \ \leq \ \sqrt{\frac{2p-3}{4}} \, \|\nabla f\|_p. \tag{7.3}$$

It is interesting that a similar statement continues to hold for the usual modulus of the gradient $|Df|$ like in the previous section within a more narrow class of functions. Namely, assume f is defined and non-negative on the cube $[0,1]^n$, is smooth and coordinatewise convex (that is, convex with respect to each coordinate). Then, whenever $f(x) \geq f(s_i(x))$, we have $f(x) - f(s_i(x)) \leq \frac{\partial f(x)}{\partial x_i}$. Hence, for any $p \geq 2$,

$$\begin{aligned} f(x)^{p/2} - f(s_i(x))^{p/2} \ &\leq \ \frac{p}{2} \, \big(f(x) - f(s_i(x))\big) \, f(x)^{(p-2)/2} \\ &\leq \ \frac{p}{2} \, \frac{\partial f(x)}{\partial x_i} \, f(x)^{(p-2)/2}, \end{aligned}$$

and so,

$$\begin{aligned}\mathbf{E}\,|\nabla f(x)^{p/2}|^2 &= \frac{2}{2^n}\sum_x \sum_{y\sim x}\big(f(x)^{p/2}-f(y)^{p/2}\big)^2\,1_{\{f(x)>f(y)\}}\\ &\le \frac{2}{2^n}\sum_x \sum_{i=1}^n \frac{p^2}{4}\,f(x)^{(p-2)}\left|\frac{\partial f(x)}{\partial x_i}\right|^2\\ &= \frac{p^2}{2}\,\mathbf{E}\,f(x)^{p-2}\,|Df(x)|^2.\end{aligned}$$

Therefore, by (7.1) and Hölder's inequality,

$$\mathrm{Ent}\big(f^p\big) \le \frac{p^2}{4}\,\mathbf{E}\,f^{p-2}\,|Df|^2 \le \frac{p^2}{4}\,\|f\|_p^{p-2}\,\|Df\|_p^2.$$

We are in a similar situation as in Section 2. By (2.5), we obtain $\frac{d}{dp}\|f\|_p^2 \le \frac{1}{2}\,\|Df\|_p^2$, which can be integrated to yield:

Corollary 7.2. *For any smooth, coordinatewise convex $f : [0,1]^n \to [0,+\infty)$, and any $p \ge 2$,*

$$\|f\|_p^2 - \|f\|_2^2 \le \frac{p-2}{2}\,\|Df\|_p^2.$$

This inequality may be used to bound probabilities of large deviations of Lipschitz coordinatewise convex functions above the mean (similarly to the approaches of [B1] and [L1]).

8. Appendix

Here we indicate how to derive the inequalities (1.5)-(1.6) and their generalizations for the class of the Orlicz norms $\|\cdot\|_{\psi_\alpha}$, generated by the Young functions $\psi_\alpha(t) = e^{|t|^\alpha} - 1$ with an arbitrary parameter $\alpha \ge 1$.

Given a measurable function f on a probability space (V,μ), first assume that $\|f\|_p \le p^{1/\alpha}$, for all $p \ge 1$. By the Taylor expansion and using the bound $n^n \le e^n\,n!$, we obtain that, for any $t \in [0, \frac{1}{\alpha e})$,

$$\mathbf{E}\,e^{t|f|^\alpha} = 1 + \sum_{n=1}^\infty \frac{t^n}{n!}\,\mathbf{E}\,|f|^{\alpha n} \le 1 + \sum_{n=1}^\infty \frac{t^n}{n!}\,(\alpha n)^n \le \frac{1}{1-\alpha e\,t}.$$

For the value $t = \frac{1}{2\alpha e}$, the ratio on the right-hand side does not exceed 2, which means that $\|f\|_{\psi_\alpha}^\alpha \leq 2\alpha e$. Hence, $\|f\|_{\psi_\alpha} \leq (2\alpha e)^{1/\alpha} \leq 2e$, since the function $\alpha \to (2\alpha e)^{1/\alpha}$ is decreasing in $\alpha \geq 1$.

Conversely, assume $\|f\|_{\psi_\alpha} \leq 1$, that is, $\mathbf{E}\, e^{|f|^\alpha} \leq 2$. Using an elementary inequality $x^p \leq Ce^{x^\alpha}$ $(x \geq 0)$ with an optimal constant $C = (\frac{p}{\alpha e})^{p/\alpha}$, we get that $\mathbf{E}\,|f|^p \leq 2C$, so, $\|f\|_p \leq 2^{1/p}\,(\frac{p}{\alpha e})^{1/\alpha} \leq 2p^{1/\alpha}$.

One can now combine the two bounds, using the homogeneity of the inequalities like (1.5)-(1.6) with respect to f.

Proposition 8.1. *Given $\alpha \geq 1$, for any measurable function f on V,*

$$\frac{1}{2}\sup_{p\geq 1}\frac{\|f\|_p}{p^{1/\alpha}} \leq \|f\|_{\psi_\alpha} \leq 2e\sup_{p\geq 1}\frac{\|f\|_p}{p^{1/\alpha}}. \tag{8.1}$$

Thus, the equality $\|f\|_{(\alpha)} = \sup_{p\geq 1}\left[p^{-1/\alpha}\,\|f\|_p\right]$ provides a family of the norms that are equivalent to $\|f\|_{\psi_\alpha}$. These norms grow with α, and $\lim_{\alpha\to\infty}\|f\|_{(\alpha)} = \|f\|_\infty$.

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